Associative algebras associated to étale groupoids and inverse semigroups

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Partial Actions and Representations Symposium

Outline

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- Morita equivalence of groupoid algebras is often explained by a Morita equivalence of the groupoids.

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- Counting measure gives a left Haar system in this context.

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- Leavitt path algebras are discrete analogues of Cuntz-Krieger algebras.
- Discrete analogues of algebras of higher rank graphs have also been considered.
- Surprising similarities between operator algebras and their discrete analogues have been known for some time.

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- My hopes have since been borne out by J. Brown, L. O. Clark, C. Farthing, A. Sims and M. Tomforde who produce new results faster than I can keep up with.

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- The sum is finite because fibers of d are discrete and f, g have compact support.

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- Leavitt path algebras can be obtained from the usual groupoid for graph C^*-algebras (see also later in the talk).

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- When $\mathcal{G}$ is Hausdorff, the kernel is generated by $\chi_U + \chi_V - \chi_{U \cup V}$ with U, V disjoint compact open subsets of $\mathcal{G}_0$

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Theorem (BS)

The class functions form the center of $\Bbbk\mathcal{G}$.

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Let $\mathcal{G}$ be a minimal, Hausdorff ample groupoid.

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2. *If $\mathcal{G}_0$ is not compact, then $Z(\Bbbk\mathcal{G}) = 0$.*

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Theorem (L. O. Clark, C. Edie-Michelle)

Let $\mathcal{G}$ be a Hausdorff ample groupoid and $\Bbbk$ a field. Then $\Bbbk\mathcal{G}$ is simple if and only if $\mathcal{G}$ is effective and minimal.

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Let $\mathcal{G}$ be a Hausdorff ample groupoid and $\Bbbk$ a field. Then $\Bbbk\mathcal{G}$ is simple if and only if $\mathcal{G}$ is effective and minimal.

- This was first proved by J.H. Brown, L.O. Clark, C. Farthing and A. Sims over $\mathbb{C}$.

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- Groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if there is a locally compact space Z and continuous open surjections $p\colon Z \to \mathcal{G}_0$ and $q\colon Z \to \mathcal{H}_0$ such that $\mathcal{G}[Z] \cong \mathcal{H}[Z]$.

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- Explains why the same moves preserve Morita equivalence between graph C^*-algebras and Leavitt path algebras.

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- If $f \in \Bbbk\mathcal{G}$ and $t \in L_x$, then

$$f \cdot t = \sum_{d(s)=r(t)} f(s)st.$$

Induction and restriction functors

- There is an exact functor $\mathrm{Ind}_x\colon \Bbbk G_x\text{-mod} \to \Bbbk\mathcal{G}\text{-mod}$ given by

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Theorem (BS)

Let $\Bbbk$ be a field and $\mathcal{G}$ an ample groupoid. Then the finite dimensional simple $\Bbbk\mathcal{G}$-modules are those of the form $\mathrm{Ind}_x(M)$ with the orbit of x finite and M a finite dimensional simple $\Bbbk G_x$-module.

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Let $\mathcal{G}$ be an effective, Hausdorff ample groupoid.

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1. *If $\Bbbk$ semiprimitive, then $\Bbbk \mathcal{G}$ is semiprimitive.*
2. *If $\Bbbk$ is a field and $\mathcal{G}$ has a dense orbit, then $\Bbbk \mathcal{G}$ is primitive.*

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- Often we will assume S has a zero element.

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- For group actions, this is the usual action groupoid.

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- This generalizes Paterson's result for C^*-algebras and my result for finite inverse semigroups.

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- χ is tight iff $\chi^{-1}(1)$ is a limit of ultrafilters.

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Theorem (BS, unpublished)

Let S be a Hausdorff inverse semigroup. Then $\Bbbk\mathcal{G}(S)_T$ is isomorphic to $\Bbbk S/I$ where I is the ideal generated by elements of the form $e - (e_1 \vee \cdots \vee e_n)$ such that $e_1, \ldots, e_n$ cover e.

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- $\Bbbk\mathcal{G}(P_X)_T$ is the Leavitt path algebra analogue of $\mathcal{O}_X$.
- More generally, Leavitt path algebras are the tight algebras of graph inverse semigroups.

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- A semigroup with no non-trivial homomorphic images is called congruence-free.

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- We recover the result that G strongly connected with no vertex of in-degree 1 implies $L_{\mathbb{k}}(G)$ is simple.
- We also recover semiprimitivity of $L_{\mathbb{k}}(G)$ over a semiprimitive base ring in the case that no vertex has in-degree 1.

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- Deal with the non-Hausdorff case.

The end

Thank you for your attention!

Obrigado pela sua atenção